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THE STABILITY OF A THIN MOVING ELASTIC STRIP SUBJECTED TO RANDOM PARAMETRIC EXCITATIONS

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1. INTRODUCTION

Several authors have recently explored transverse motion stability of a thin moving elastic strip subjected to a plane deterministic or random load. This problem has been analyzed in the papers of Soler [1], Tylikowsky [2] and Kozin and Milstead [3]. They have used various models for deriving equations of the strip motion as well as for external loading. Their aim was to determine the motion stability conditions or to define the system parameter's impact upon frequencies of various mechanical systems such as band saws, transmission belts or strip motion in modern instrumentation tape recorders. In reference [1] the deterministic problem has been discussed and in addition the dependence of frequency upon the system parameters has been determined. In reference [2] uniform stochastic stability has been discussed by using the Lyapunov direct method. In reference [3] the method developed by Wu and Kozin was applied in order to determine the conditions for an almost-sure asymptotic system stability of an order greater than the second.

In what follows, combining the method of stochastic averaging and the procedure for determining the largest Lyapunov exponent, the largest Lyapunov exponent has been defined for the mechanical system concerned when subjected to random non-correlative parametric excitations. On the basis of these expressions and provided that the largest Lyapunov exponent is negative, an asymptotic stability region has been determined for the case when $P(t) = 0$ for various system parameters. The mean square stability region has also been determined.

2. FORMULATION OF THE PROBLEM

In reference [1] the nondimensional differential equations of motion of a moving thin elastic strip (see Figure 1) were derived under the following conditions.

It is assumed that displacements in the co-ordinate directions are small so that the classical von Kármán formulation is used; additionally, it is assumed that the transverse displacement in the z direction is sufficiently small so that the strip's cross-section deflects and rotates like a rigid body. The effect of the gyroscopic terms is neglected.

$$\gamma^2\left(\frac{b}{L}\right)^2\frac{\partial^4 u}{\partial\xi^4} + [(\mu^2 - \zeta^2) - \eta_1(\tau)]\frac{\partial^2 u}{\partial\xi^2} + 2\mu\frac{\partial^2 u}{\partial\xi\partial\tau} + \frac{\partial^2 u}{\partial\tau^2} + \beta'\left[\frac{\partial u}{\partial\tau} + \mu\frac{\partial u}{\partial\xi}\right]$$

$$- [\bar{\eta}_2 + \eta_2(\tau)]\frac{\partial^2\varphi}{\partial\xi^2} + \left(\frac{L}{h}\right)[\bar{\eta}_3 + \eta_3(\tau)]\frac{\partial\varphi}{\partial\xi} = 0, \tag{1a}$$

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$$\gamma^2\left(\frac{b}{L}\right)^2\frac{\partial^4\varphi}{\partial\xi^4}+[(\mu^2-1)-\eta_1(\tau)]\frac{\partial^2\varphi}{\partial\xi^2}+2\mu\frac{\partial^2\varphi}{\partial\xi\partial\tau}+\frac{\partial^2\varphi}{\partial\tau^2}+\beta'\left[\frac{\partial\varphi}{\partial\tau}+\mu\frac{\partial\varphi}{\partial\xi}\right]$$

$$-[\bar{\eta}_2+\eta_2(\tau)]\frac{\partial^2u}{\partial\xi^2}+\left(\frac{L}{h}\right)[\bar{\eta}_3+\eta_3(\tau)]\frac{\partial u}{\partial\xi}=0 \qquad (1b)$$

Here the following parameters have been introduced for the sake of simplifying the expression as well as for forming the non-dimensional equations of motion:

$$u=\frac{W}{(12)^{1/2}b},\qquad \varphi=\frac{h\theta}{(12)^{1/2}b},\qquad \xi=\frac{x}{L},\qquad C_T^2=\frac{4Gb^2}{\rho h^2}\left[1+\frac{T_0h^2}{4Gb^2}\right],\qquad \varsigma^2=\frac{T_0}{\rho bhC_T^2},$$

$$\tau=\frac{tC_T}{L},\qquad \eta_1(\tau)=\frac{T(t)}{\rho bhC_T^2},\qquad \bar{\eta}_2=-\frac{\xi P_0}{\rho bhC_T^2}\left(\frac{L}{h}\right),\qquad \eta_2(\tau)=-\frac{\xi P(\tau)}{\rho bhC_T^2}\left(\frac{L}{h}\right),$$

$$\bar{\eta}_3=\frac{P_0}{\rho bhC_T^2},\qquad \eta_3(\tau)=\frac{P(\tau)}{\rho bhC_T^2},\qquad \mu=\frac{C}{C_T},\qquad 2\varepsilon\beta'=\frac{\beta L}{\rho bC_T},\qquad \gamma^2=\frac{E}{12\rho(1-v^2)C_T^2}.$$

Here: E is the Young's modulus of the strip material, G is the shear modulus of the strip material, ρ is the strip's mass density, W is the transverse displacement (z direction), θ is the rotation of the strip about the x-axis, C is the constant axial (x direction) translation velocity, v is the Poisson's ratio of the strip, T_0 is the constant tension in the strip, P_0 is the constant edge load at the support, $T(\tau)$ and $P(\tau)$ are stationary stochastic wide-band non-correlated processes of small intensity with zero mean value, b is the strip thickness, h is the strip width, L is the separation between pinch rollers or drive capstans, and $\beta > 0$ is the small coefficient of linear damping.

In order to simplify equations (1a, 1b) Galerkin's method has been used to reduce them to an equivalent system of ordinary differential equations with respect to the time-dependent parts of the solution as shown in reference [3]:

$$\ddot{U}_n+2\varepsilon\beta'\dot{U}_n+[C_1^{(n)}+\varepsilon^{1/2}f_{(\tau)}^{(n)}]\,U_n+[C_2^{(n)}+\varepsilon^{1/2}g_{(\tau)}^{(n)}]\,V_n=0 \qquad (2)$$

$$\ddot{V}_n+2\varepsilon\beta'\dot{V}_n+[C_2^{(n)}+\varepsilon^{1/2}g_{(\tau)}^{(n)}]\,U_n+[C_3^{(n)}+\varepsilon^{1/2}f_{(\tau)}^{(n)}]\,V_n=0 \qquad (3)$$

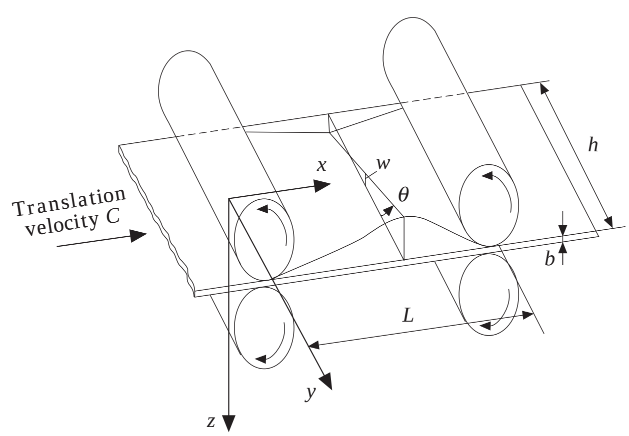

Figure 1. Sketch and geometry of the moving strip.

Here,

$$C_1^{(n)} = (n\pi)^4\left[\gamma^2\left(\frac{b}{L}\right)^2 + \frac{\zeta^2 - \mu^2}{(n\pi)^2}\right], \qquad C_3^{(n)} = (n\pi)^4\left[\gamma^2\left(\frac{b}{L}\right)^2 + \frac{1 - \mu^2}{(n\pi)^2}\right],$$

$$C_2^{(n)} = (n\pi)^2\zeta^2\left(\frac{L}{2h}\right)^2\frac{P_0}{T_0}, \qquad \varepsilon^{1/2}f_{(\tau)}^{(n)} = (n\pi)^2\zeta^2\frac{T_{(\tau)}}{T_0}, \qquad \varepsilon^{1/2}g_{(\tau)}^{(n)} = (n\pi)^2\zeta^2\left(\frac{L}{2h}\right)\frac{P_{(\tau)}}{T_0}.$$

3. TRANSFORMATION TO "STANDARD FORM"

The eigenfrequencies $\omega_1^{(n)}$, $\omega_2^{(n)}$ of the unperturbed system ($\varepsilon = 0$) are given by the roots of the equation

$$[\omega^{(n)}]^4 - [C_1^{(n)} + C_3^{(n)}][\omega^{(n)}]^2 + C_1^{(n)}C_3^{(n)} - [C_2^{(n)}]^2 = 0, \tag{4}$$

or

$$[\omega_{1/2}^{(n)}]^2 = (C_1^{(n)} + C_3^{(n)} \pm \sqrt{[C_1^{(n)} - C_3^{(n)}]^2 + 4[C_2^{(n)}]^2})/2. \tag{5}$$

The transformation of equations (2, 3) to the "standard form" is done by using the relations

$$U_n = a_1\cos\theta_1 + a_2\cos\theta_2, \qquad V_n = \alpha_1 a_1\cos\theta_1 + \alpha_2 a_2\cos\theta_2,$$

where

$$\theta_1 = \omega_1^{(n)}\tau + \varphi_1, \qquad \theta_2 = \omega_2^{(n)}\tau + \varphi_2, \qquad \alpha_1 = \frac{[\omega_1^{(n)}]^2 - C_1^{(n)}}{C_2^{(n)}} = \frac{C_2^{(n)}}{[\omega_1^{(n)}]^2 - C_3^{(n)}},$$

$$\alpha_2 = \frac{[\omega_2^{(n)}]^2 - C_1^{(n)}}{C_2^{(n)}} = \frac{C_2^{(n)}}{[\omega_2^{(n)}]^2 - C_3^{(n)}}.$$

By the well-known procedure, an equivalent system is obtained consisting of four first order equations of the "standard form";

$$\frac{\mathrm{d}a_1}{\mathrm{d}\tau} = A\omega_2^{(n)}\{-2\varepsilon\beta' a_1\omega_1^{(n)}(\alpha_1 - \alpha_2)\sin^2\theta_1 + \varepsilon^{1/2}f_{(\tau)}^{(n)}(\alpha_1 - \alpha_2)a_1\sin\theta_1\cos\theta_1$$
$$+ \varepsilon^{1/2}g_{(\tau)}^{(n)}[(1 - \alpha_1\alpha_2)a_1\sin\theta_1\cos\theta_1 + (1 - \alpha_2^2)a_2\cos\theta_2\sin\theta_1]\}, \tag{6a}$$

$$\frac{\mathrm{d}a_2}{\mathrm{d}\tau} = A\omega_1^{(n)}\{-2\varepsilon\beta' a_2\omega_2^{(n)}(\alpha_1 - \alpha_2)\sin^2\theta_2 + \varepsilon^{1/2}f_{(\tau)}^{(n)}(\alpha_1 - \alpha_2)a_2\sin\theta_2\cos\theta_2$$
$$- \varepsilon^{1/2}g_{(\tau)}^{(n)}[(1 - \alpha_1\alpha_2)a_2\sin\theta_2\cos\theta_2 + (1 - \alpha_1^2)a_1\cos\theta_1\sin\theta_2]\}, \tag{6b}$$

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}\tau} = \frac{A\omega_2^{(n)}}{a_1}\{-2\varepsilon\beta' a_1\omega_1^{(n)}(\alpha_1 - \alpha_2)\sin\theta_1\cos\theta_1 + \varepsilon^{1/2}f_{(\tau)}^{(n)}(\alpha_1 - \alpha_2)a_1\cos^2\theta_1$$
$$+ \varepsilon^{1/2}g_{(\tau)}^{(n)}[(1 - \alpha_1\alpha_2)a_1\cos^2\theta_1 + (1 - \alpha_2^2)a_2\cos\theta_1\cos\theta_2]\}, \tag{6c}$$

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}\tau} = \frac{A\omega_1^{(n)}}{a_2}\{-2\varepsilon\beta' a_2\omega_2^{(n)}(\alpha_1 - \alpha_2)\sin\theta_2\cos\theta_2 + \varepsilon^{1/2}f_{(\tau)}^{(n)}(\alpha_1 - \alpha_2)a_2\cos^2\theta_2$$
$$- \varepsilon^{1/2}g_{(\tau)}^{(n)}[(1 - \alpha_1\alpha_2)a_2\cos^2\theta_2 + (1 - \alpha_1^2)a_1\cos\theta_2\cos\theta_1]\}. \tag{6d}$$

Here,

$$A = 1/\omega_1^{(n)}\omega_2^{(n)}(\alpha_1 - \alpha_2).$$

4. LYAPUNOV EXPONENT

By applying the procedure of stochastic averaging [6] to equations (6a–d) one obtains the following Itô differential equations with respect to the averaged amplitudes a_1 and a_2, in the first approximation:

$$\mathrm{d}a_1 = \varepsilon K_1\,\mathrm{d}\tau + \varepsilon^{1/2}\sigma_{11}\,\mathrm{d}W_1 + \varepsilon^{1/2}\sigma_{12}\,\mathrm{d}W_2 \tag{7}$$

$$\mathrm{d}a_2 = \varepsilon K_2\,\mathrm{d}\tau + \varepsilon^{1/2}\sigma_{21}\,\mathrm{d}W_1 + \varepsilon^{1/2}\sigma_{22}\,\mathrm{d}W_2. \tag{8}$$

Here $W_1(\tau)$ and $W_2(\tau)$ are independent Wiener processes of unit intensity while the drift K_1, K_2 and the diffusion coefficients σ_{ij}, $i, j = 1, 2$ are given by the following relations:

$$K_1 = -\beta'\omega_1^{(n)}\omega_2^{(n)}(\alpha_1-\alpha_2)Aa_1 + \{\tfrac{3}{16}(\alpha_1-\alpha_2)^2a_1S_f + \tfrac{1}{16}[3(1-\alpha_1\alpha_2)^2a_1 + 2(1-\alpha_2^2)^2a_2^2/a_1]S_g\}A^2[\omega_2^{(n)}]^2,$$

$$K_2 = -\beta'\omega_1^{(n)}\omega_2^{(n)}(\alpha_1-\alpha_2)Aa_2 + \{\tfrac{3}{16}(\alpha_1-\alpha_2)^2a_2S_f + \tfrac{1}{16}[3(1-\alpha_1\alpha_2)^2a_2 + 2(1-\alpha_1^2)^2a_1^2/a_2]S_g\}A^2[\omega_1^{(n)}]^2,$$

$$[\sigma\sigma^{\mathrm{T}}]_{11} = \tfrac{1}{8}\{(\alpha_1-\alpha_2)^2a_1^2S_f + [(1-\alpha_1\alpha_2)^2a_1^2 + 2(1-\alpha_2^2)^2a_2^2]S_g\}A^2[\omega_2^{(n)}]^2,$$

$$[\sigma\sigma^{\mathrm{T}}]_{22} = \tfrac{1}{8}\{(\alpha_1-\alpha_2)^2a_2^2S_f + [(1-\alpha_1\alpha_2)^2a_2^2 + 2(1-\alpha_1^2)^2a_1^2]S_g\}A^2[\omega_1^{(n)}]^2,$$

$$[\sigma\sigma^{\mathrm{T}}]_{12} = 0, \qquad [\sigma\sigma^{\mathrm{T}}]_{21} = 0.$$

Spectral densities S_f and S_g of the processes $f_{(\tau)}^{(n)}$ and $g_{(\tau)}^{(n)}$ are defined by the relations

$$S_f = 2\int_0^\infty \mathrm{E}[f_{(\tau)}^{(n)}f_{(\tau+t)}^{(n)}]\cos\omega t\,\mathrm{d}t, \qquad S_g = 2\int_0^\infty \mathrm{E}[g_{(\tau)}^{(n)}g_{(\tau+t)}^{(n)}]\cos\omega t\,\mathrm{d}t,$$

where E[] denotes the expectation.

One can notice that the averaged amplitude vector (a_1, a_2) is a two-dimensional diffusion process while the coefficients on the right side of equations (7, 8) are homogeneous with respect to the first degrees of the amplitudes a_1 and a_2. In order to apply the Khas'minskii transformation and to derive the expression for the largest Lyapunov exponent the logarithmic polar transformation is introduced:

$$\rho = \tfrac{1}{2}\log(a_1^2 + a_2^2), \qquad \tan\phi = a_2/a_1,\ 0 \leqslant \phi \leqslant \pi/2.$$

By the procedure given in references [4, 5] the expression for the largest Lyapunov exponent are obtained as

$$\lambda = \tfrac{1}{2}\{(\lambda_1+\lambda_2) + (\lambda_1-\lambda_2)\coth[(\lambda_1-\lambda_2)\alpha/\Delta^{1/2}]\}, \quad \Delta > 0, \tag{9a}$$

where α is given by

$$\tanh\alpha = \left[1 - \left(\frac{4k_{12}k_{21}}{[k_{11}^2 + k_{22}^2]}\right)^2\right]^{1/2}.$$

For the case when $\Delta < 0$ the Lyapunov exponent is

$$\lambda = \tfrac{1}{2}[(\lambda_1+\lambda_2) + (\lambda_1-\lambda_2)\coth[\lambda_1-\lambda_2)\alpha/(-\Delta)^{1/2}]\}, \qquad \Delta < 0, \tag{9b}$$

where α is given by

$$\tan\alpha = \left[\left(\frac{4k_{12}k_{21}}{[k_{11}^2 + k_{22}^2]}\right)^2 - 1\right]^{1/2}.$$

In the case when $\Delta = 0$, Lyapunov exponent is

$$\lambda = \tfrac{1}{2}\{(\lambda_1 + \lambda_2) + (\lambda_1 - \lambda_2)\coth[2(\lambda_1 - \lambda_2)/|k_{12}k_{21}|]\}, \tag{9c}$$

in which the constants k_{11}, k_{12}, k_{21}, k_{22}, λ_1, λ_2 and Δ are defined by the relations

$$k_{11}^2 = A^2[\omega_2^{(n)}]^2[(\alpha_1 - \alpha_2)^2 S_f + (1 - \alpha_1\alpha_2)^2 S_g], \qquad k_{12}^2 = A^2[\omega_2^{(n)}]^2(1 - \alpha_2^2)^2 S_g,$$

$$k_{21}^2 = A^2[\omega_1^{(n)}]^2[(1 - \alpha_1^2)^2 S_g, \qquad k_{22}^2 = A^2[\omega_1^{(n)}]^2[(\alpha_1 - \alpha_2)^2 S_f + (1 - \alpha_1\alpha_2)^2 S_g],$$

$$\lambda_1 = -\beta' + k_{11}^2/8, \qquad \lambda_2 = -\beta' + k_{22}^2/8,$$

$$\Delta = b^2 - 4ac = \tfrac{1}{64}[(k_{11}^2 + k_{22}^2)^2 - 16k_{12}^2k_{21}^2].$$

5. STABILITY REGIONS FOR A MOVING ELASTIC STRIP

By using the expressions (9a–c) for real numerical system parameters, from the condition that the largest Lyapunov exponent is negative, it is theoretically possible to determine an almost-sure region of asymptotic stability, with the probability of one, of the mechanical system under study. In this case this is a very complex numerical problem. Consider the case when the load on the support is of constant intensity: that is when $P(t) = 0$, $S_g = 0$. Then the constants are

$$k_{11}^2 = A^2[\omega_2^{(n)}]^2(\alpha_1 - \alpha_2)^2 S_f, \qquad k_{12}^2 = 0, \qquad \lambda_1 = -\beta' + k_{11}^2/8,$$

$$k_{21}^2 = 0, \qquad k_{22}^2 = A^2[\omega_1^{(n)}]^2(\alpha_1 - \alpha_2)^2 S_f, \qquad \lambda_2 = -\beta' + k_{22}^2/8,$$

$$\Delta = b^2 - 4ac = \tfrac{1}{64}(k_{11}^2 + k_{22}^2)^2 > 0.$$

For the values of the constant $\Delta > 0$ the stability region boundaries are determined by the expression (9a), from which it follows that $\alpha \to \infty$ and

$$\lambda_1 = -\beta' + k_{11}^2/8 < 0. \tag{10}$$

By substituting the constants in expression (10) the condition for an almost-sure asymptotic stability is found to be,

$$\beta' > (1/8[\omega_1^{(n)}]^2)S_f. \tag{11}$$

By using the Itô rule for differentiating complex stochastic functions one can simply determine the differential equations in the second order moment of the amplitudes, and take the expectations of both sides of the resulting equations, yielding

$$\mathrm{dE}\,[a_1^2]/\mathrm{d}\tau = \varepsilon[-2\beta' + S_f/2[\omega_1^{(n)}]^2]\,\mathrm{E}\,[a_1^2]. \tag{12}$$

$$\mathrm{dE}\,[a_2^2]/\mathrm{d}\tau = \varepsilon[-2\beta' + S_f/2[\omega_2^{(n)}]^2]\,\mathrm{E}\,[a_2^2]. \tag{13}$$

Therefore, on the basis of equations (12, 13) the necessary and sufficient condition for mean square stability is

$$\beta' > (1/4[\omega_1^{(n)}]^2)S_f \tag{14}$$

The variations of the stable regions almost-sure asymptotic stability from expression (11) and mean-square stability from expression (14), for different values of the squared eigenfrequency $[\omega_1^{(n)}]^2 = 100$, $150\ \mathrm{s}^{-2}$, are shown in Figure 2. Notice that the stability region

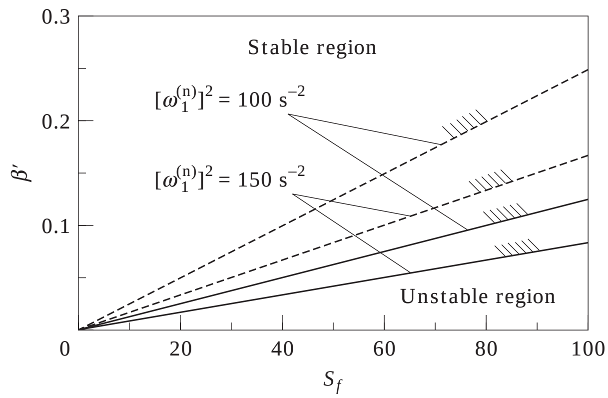

Figure 2. Stability regions for a moving elastic strip. - - - -, Mean-square stability; —, almost-sure asymptotic stability.

of the mean-square amplitudes is twice as large as the stability region of almost-sure asymptotic stability for both values of the square eigenfrequency $[\omega_1^{(n)}]^2$.

6. CONCLUSIONS

Stochastic stability of a thin moving elastic strip when it is subjected to parameters of random excitations that are wide-band stochastic processes of small intensity has been examined. By combining the Khas'minskii method of stochastic averaging and the procedure for determining the largest Lyapunov exponent, expressions for the largest Lyapunov exponent with respect to the system parameters have been determined. Especially for the case $P_{(t)} = 0$, by using the derived expressions an almost-sure asymptotic stability region of the mechanical system has been determined. The mean-square stability region has also been determined. One can also notice that with the higher eigenfrequency of the system one has the higher stability region as determined for both criteria.

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